

INFINITESIMAL CONFORMAL DEFORMATIONS OF TRIANGULATED SURFACES IN SPACE

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ABSTRACT. We study the change in the extrinsic geometry of a triangulated surface under infinitesimal conformal deformations in Euclidean space. A deformation of vertices is conformal if it preserves length cross-ratios. On the one hand, conformal deformations generalize deformations preserving edge lengths. On the other hand, there is a one-to-one correspondence between infinitesimal conformal deformations in space and infinitesimal isometric deformations of the stereographic image on the sphere. We parametrize the space of infinitesimal conformal deformations in terms of the change in dihedral angles, which is closely related to the Schläfli formula.

1. INTRODUCTION

Realizing a triangulated surface in Euclidean space with prescribed edge lengths is a classical problem in rigidity theory [3]. Fixing a combinatorial structure and a discrete metric, one is interested in determining the existence and uniqueness of the realization, which is analogous to the problem of isometric immersions of surfaces in differential geometry. It stimulates various directions of research, such as infinitesimal rigidity. A triangulated surface in space is infinitesimally rigid if all its first-order isometric deformations are induced by Euclidean motions. Dehn's rigidity theorem [5] states that all convex polyhedra are infinitesimally rigid. Gluck [7] further shows that generic triangulated spheres are infinitesimally rigid.

Rather than insisting on edges lengths, we are interested in infinitesimal deformations of triangulated surfaces preserving conformal structures.

The concept of discrete conformality arose from William Thurston's idea to approximate conformal maps by circle packings in the plane [25]. Rodin and Sullivan [21] proved the convergence of the analogue of Riemann maps for circle packings. There are further extensions, such as Schramm's orthogonal circle patterns [23], where circles are allowed to intercept each other. The intersection angles of circles yield a discrete notion of conformal structure. Such conformal structure is well-defined in Möbius geometry since Möbius transformations map circles to circles and preserve their intersection angles.

For a given triangulated surface in space, one might be tempted to define its conformal structure in terms of the intersection angles of the circumscribed circles. However, deformations preserving edge lengths generally do not preserve the intersection angles because of the change in dihedral angles. This phenomenon is dissatisfying since it is not consistent with the smooth theory, where isometric deformations are special cases of conformal deformations. In fact, Bobenko and Schröder [2] used the intersection angles to define an analogue of the Willmore energy instead of a conformal structure. On the other hand, in an effort to remedy the problem, one might measure the intersection angles after flattening neighboring triangles into the plane so as to remove the dependence on dihedral angles [12]. Nevertheless, the angles measured in this way will vary under Möbius transformations.

As a counterpart for circle patterns, another notion of discrete conformality, length cross-ratios, was proposed by Luo [20, 16, 24]. For a triangle mesh in the plane, one can associate

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a complex number to each interior edge by taking the cross-ratio of the four vertices of the adjacent triangles. The modulus of the cross-ratio is called the length cross-ratio, which can be expressed solely in terms of edge lengths.

Definition 1.1. *The length cross-ratio $lcr : E_{int} \rightarrow \mathbb{R}$ of a triangulated surface equipped with a discrete metric $\ell : E \rightarrow \mathbb{R}_{>0}$ is*

$$lcr_{ij} = \frac{\ell_{il}\ell_{jk}}{\ell_{lj}\ell_{ki}}$$

where $\{ijk\}, \{jil\}$ are the left and the right triangles of the oriented edge e_{ij} . Two discrete metrics are conformally equivalent if their length cross-ratios are identical.

This notion is a counterpart for circle patterns in the plane since the other half of the cross-ratio, namely the argument, yields the intersection angle of the circumscribed circles. In the plane, the corresponding infinitesimal deformations of the two types are simply related by a $\pi/2$ -rotation [14].

Previous study of the theory of length cross-ratios is restricted to intrinsic geometry. Luo [16] introduced vertex-scaling as an equivalent form of the length cross-ratio theory to study combinatorial Yamabe flow. Bobenko, Pinkall and Springborn [1] further established its relation to ideal hyperbolic polyhedra.

Conformal deformations are interesting not only from the theoretical point of view but also for applications. Numerical approximations for conformal deformations of smooth surfaces have been obtained by directly discretizing equations from the smooth theory. Gu and Yau [9] studied conformal parametrizations of triangulated surfaces. Conformal deformations of triangulated surfaces with respect to extrinsic geometry have been considered numerically [4].

In this paper, we focus on infinitesimal conformal deformations of triangulated surfaces in space in the sense of length cross-ratio theory. We show that infinitesimal conformal and isometric deformations are closely related.

Theorem 1.2. *Given a non-degenerate realization $f : V \rightarrow \mathbb{R}^n$ of a triangulated surface, the space of infinitesimal conformal deformations of f in \mathbb{R}^n is isomorphic to the space of infinitesimal isometric deformations of $\sigma \circ f$ in \mathbb{R}^{n+1} . Here $\sigma : \mathbb{R}^n \rightarrow S^n$ is the stereographic projection.*

We then study infinitesimal conformal deformations in \mathbb{R}^3 and the change in extrinsic geometry. Trivial infinitesimal conformal deformations are induced by Möbius transformations. Unlike isometric deformations, all triangulated surfaces except tetrahedra admit non-trivial infinitesimal conformal deformations. This can be simply seen by counting: A closed triangulated surface in \mathbb{R}^3 of genus g with V vertices has $3V$ degrees of freedom. Notice that the length cross-ratios satisfy $\prod_j lcr_{ij} = 1$ for every vertex i . In order to preserve the conformal structure $lcr : E \rightarrow \mathbb{R}$ infinitesimally, there are $E - V = 2V - 6 + 6g$ linear constraints. Hence the space of infinitesimal conformal deformations is at least $3V - (E - V) = V + 6 - 6g$.

Theorem 1.3. *Let $\dot{f} : V \rightarrow \mathbb{R}^3$ be an infinitesimal conformal deformation of a non-degenerate triangulated surface $f : V \rightarrow \mathbb{R}^3$ with infinitesimal scaling $u : V \rightarrow \mathbb{R}$. Then there exists a unique $Z : F \rightarrow \mathbb{R}^3$ such that*

$$\begin{aligned} d\dot{f}(e_{ij}) &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times \left(Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk} \right) \\ (1) \quad &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times \left(Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil} \right) \end{aligned}$$

where $\{ijk\}, \{jil\}$ are the left and the right faces of the oriented edge e_{ij} . The functions u, Z satisfy

$$(2) \quad -df(e_{ij}) \times dZ(e_{ij}^*) + \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle ilj}{2} N_{ilj} \right) \times df(e_{ij}) du(e_{ij}) = 0$$

and the change in dihedral angles $\dot{\alpha}$ is given by

$$\langle df(e_{ij}), dZ(e_{ij}^*) \rangle = \dot{\alpha}_{ij} |df(e_{ij})|.$$

Conversely, if the triangulated surface is simply connected and functions $u : V \rightarrow \mathbb{R}$, $Z : F \rightarrow \mathbb{R}^3$ satisfy (2), then there exists an infinitesimal conformal deformation \dot{f} satisfying (1) unique up to translations.

Theorem 1.3 enables us to relate infinitesimal conformal deformations to the change in dihedral angles. Since we are interested in the change in extrinsic geometry, a natural question is to find an infinitesimal conformal deformation with prescribed change in dihedral angles. In general this will be impossible, since the number of dihedral angles is $|E|$, which is larger than the dimension of the space of infinitesimal conformal deformations. Hence, instead of dihedral angles, we study the *change in mean curvature half-density* $\rho : V_{int} \rightarrow \mathbb{R}$,

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|.$$

We introduce a discrete Dirac operator D (Definition 7.4) to relate infinitesimal conformal deformations to the change in mean curvature half-density.

Theorem 1.4. *Suppose a closed triangulated sphere does not possess any non-trivial infinitesimal conformal deformation with vanishing change in mean curvature half-density, which means $\dim \ker D = 4$. Then, given any $\rho : V \rightarrow \mathbb{R}$ with $\sum_i \rho_i = 0$, there exists an infinitesimal conformal deformation unique up to similarities, such that*

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|$$

where $\dot{\alpha}$ is the change in dihedral angles.

Here the condition $\sum_i \rho_i = 0$ is a consequence of the Schläfli formula (Proposition 7.1).

It has been shown [14] that infinitesimal conformal deformations of triangulated disks in the plane are closely related to discrete complex analysis. This follows immediately from Theorem 1.3. Each infinitesimal conformal deformation corresponds to a discrete harmonic function with respect to the cotangent Laplacian (Corollary 6.2). The study of the planar case has led to a unified theory of discrete minimal surfaces [13].

Combining our new results with earlier ones in [15], an interesting analogy between isometric deformations and conformal deformations becomes visible (Table 1). Euclidean motions induce trivial isometric deformations. Under projective transformations, there is a one-to-one correspondence between infinitesimal isometric deformations of a triangulated surface to those of its image and hence the class of infinitesimally flexible surfaces is preserved. On the other hand, Möbius transformations induce trivial conformal deformations. Under Möbius transformations, there is also a one-to-one correspondence of infinitesimal conformal deformations. Earlier, we have studied “conformally flexible” surfaces and called them *isothermic triangulated surfaces* [15]. More precisely, a closed triangulated surface in space is isothermic if its space of infinitesimal conformal deformations has dimension strictly larger than $V + 6 - 6g$. The class of isothermic triangulated surfaces is preserved under Möbius transformations.

Our approach is motivated by the method of quaternionic analysis in the smooth theory, which relates conformal deformations to mean curvature [18]. In the smooth theory, a pair of non-congruent surfaces is a Bonnet pair if they are isometric with identical mean curvature. Using

TABLE 1. Comparison between infinitesimal isometric and conformal deformations

Types of infinitesimal deformations:	Isometric	Conformal
Constraints:	Edge lengths	Length cross-ratios
Trivial deformations:	Euclidean transformations	Möbius transformations
Singularity:	Infinitesimally flexible surfaces	Isothermic surfaces
Bijection under:	Projective transformations	Möbius transformations

quaternionic analysis, there is an elegant way to obtain Bonnet pairs from an isothermic surface [11].

In Section 3 we review the theory of length cross-ratios. We then prove Theorem 1.2 in Section 4 and Theorem 1.3 in Section 5. An immediate corollary is the angular velocity equation in Section 6. In order to relate infinitesimal conformal deformations to the change in mean curvature half-density as in Theorem 1.4, a discrete Dirac operator is derived in Section 7. Examples are given in Section 8. In Section 9, we extend the results to surfaces of high genus. Finally, the relation to isothermic surfaces and open problems are discussed in Section 10.

2. NOTATION

Definition 2.1. A triangulated surface $M = (V, E, F)$ is a finite simplicial complex whose underlying topological space is a connected 2-manifold with boundary. The set of vertices (0-cells), edges (1-cells) and triangles (2-cells) are denoted as V , E and F .

Definition 2.2. A realization of a triangulated surface in \mathbb{R}^n is a map $f : V \rightarrow \mathbb{R}^n$ which is linear on each face. We say f is non-degenerate if every face of f spans an affine 2-plane, which implies $f_i \neq f_j$ for every edge $\{ij\} \in E$.

Without further notice all triangulated surfaces under consideration are assumed to be oriented and the realizations are non-degenerate.

We denote V_{int} and E_{int} the set of interior vertices and the set of interior edges respectively. We write e_{ij} as the oriented edge from the vertex i to the vertex j . Note that $e_{ij} \neq e_{ji}$. The set of oriented edges is denoted by \vec{E} . The set of interior oriented edges is indicated by \vec{E}_{int} .

We make use of discrete differential forms from Discrete Exterior Calculus [6]. Given a triangulated surface $M = (V, E, F)$, we denote \vec{E} the set of oriented edges and \vec{E}_{int} the set of interior oriented edges. An oriented edge from vertex i to vertex j is indicated by e_{ij} . A function $\omega : \vec{E} \rightarrow \mathbb{R}$ is called a (primal) *discrete 1-form* if

$$\omega(e_{ij}) = -\omega(e_{ji}) \quad \forall e_{ij} \in \vec{E}.$$

It is *closed* if for every face $\{ijk\}$

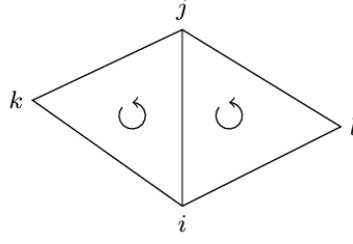
$$\omega(e_{ij}) + \omega(e_{jk}) + \omega(e_{ki}) = 0.$$

It is *exact* if there exists a function $f : V \rightarrow \mathbb{R}$ such that for $e_{ij} \in \vec{E}$

$$\omega(e_{ij}) = f_j - f_i =: df(e_{ij}).$$

Similarly, we consider discrete 1-forms on the dual mesh M^* and these are called dual 1-forms on M . For every oriented edge e , we write e^* as its dual edge oriented from the right face of e to its left face. A function $\eta : \vec{E}_{int}^* \rightarrow \mathbb{R}$ defined on oriented dual edges is called a dual 1-form if

$$\eta(e_{ij}^*) = -\eta(e_{ji}^*) \quad \forall e_{ij}^* \in \vec{E}_{int}^*.$$


 FIGURE 1. Two neighboring triangles containing edge e_{ij}

A dual 1-form η is *closed* if for ever interior vertex $i \in V_{int}$

$$\sum_j \eta(e_{ij}^*) = 0.$$

It is exact if there exists $h : F \rightarrow \mathbb{R}$ such that

$$df(e_{ij}^*) := h_{ijk} - h_{jil} = \eta(e_{ij}^*)$$

where $\{ijk\}$ denotes the left face of e_{ij} and $\{jil\}$ denotes the right face (Figure 1).

Give a non-degenerate realization $f : V \rightarrow \mathbb{R}^3$ of a triangulated surface, we measure its dihedral angles $\alpha : E_{int} \rightarrow \mathbb{R}$. Denoting $N : F \rightarrow \mathbb{S}^2$ the face normal compatible with the orientation, the sign of the dihedral angle $\alpha_{ij} \in (-\pi, \pi)$ is determined by

$$\begin{aligned} \sin \alpha_{ij} &= \langle N_{ijk} \times N_{jil}, \frac{df(e_{ij})}{|df(e_{ij})|} \rangle, \\ \cos \alpha_{ij} &= \langle N_{ijk}, N_{jil} \rangle \end{aligned}$$

where $\{ijk\}, \{jil\} \in F$ denote the left and the right face of e_{ij} .

3. CONFORMAL EQUIVALENCE

This section reviews the definition of the conformal equivalence of triangulated surfaces based on length cross-ratios [16, 24], which possesses properties as in the smooth theory. Every immersion of a triangulated surface into Euclidean space induces a conformal structure. The conformal structure is preserved under Möbius transformations of vertices. Bobenko, Pinkall and Springborn [1] showed that every conformal structure associates a complete hyperbolic metric on a punctured surface.

Definition 3.1. A discrete metric on a triangulated surface is a function $\ell : E \rightarrow \mathbb{R}_{>0}$ satisfying the triangle inequality. Two discrete metrics on a triangulated surface are isometric if they are identical.

In Definition 1.1, we say two triangulated surfaces are conformally equivalent if their length cross-ratios are the same. It is equivalent to say that their discrete metrics are related by vertex scaling.

Proposition 3.2 ([1]). Two discrete metrics $\ell, \tilde{\ell} : E \rightarrow \mathbb{R}_{>0}$ are conformally equivalent if and only if there exists $u : V \rightarrow \mathbb{R}$ such that for every edge $\{ij\}$

$$\tilde{\ell}_{ij} = e^{\frac{u_i + u_j}{2}} \ell_{ij}.$$

We are interested in discrete metrics induced from non-degenerate realizations into \mathbb{R}^n .

Definition 3.3. Every non-degenerate realization $f : V \rightarrow \mathbb{R}^n$ of a triangulated surface induces a discrete metric $\ell : E \rightarrow \mathbb{R}_{>0}$

$$\ell_{ij} = |f_j - f_i| \quad \forall \{ij\} \in E.$$

Two non-degenerate realizations are conformally equivalent if their induced edge lengths are conformally equivalent.

Conformally equivalent classes are invariant under Möbius transformations, in contrast to the Euclidean invariance of isometry classes.

Proposition 3.4. Suppose $f : V \rightarrow \mathbb{R}^n$ is a non-degenerate realization of a triangulated surface. Then for any Möbius transformation ϕ of $\mathbb{R}^n \cup \{\infty\}$, the realizations f and $\phi \circ f$ are conformally equivalent.

Proof. Möbius transformations are generated by translations, dilations and the inversion under the unit sphere. Conformal equivalence is preserved obviously under dilations and translations. Thus, it suffices to consider ϕ as the inversion under the unit sphere. Since

$$\begin{aligned} \left\| \frac{f_i}{|f_i|^2} - \frac{f_j}{|f_j|^2} \right\|^2 &= \frac{1}{|f_i|^2} + \frac{1}{|f_j|^2} - \frac{2}{|f_i|^2 |f_j|^2} \langle f_i, f_j \rangle \\ &= \frac{1}{|f_i|^2 |f_j|^2} \|f_i - f_j\|^2 \end{aligned}$$

we conclude f and $\phi \circ f$ are conformally equivalent. \square

The following is a linearization of vertex scaling (Proposition 3.2).

Definition 3.5. Suppose $f : V \rightarrow \mathbb{R}^n$ is a non-degenerate realization of a triangulated surface. An infinitesimal deformation $\dot{f} : V \rightarrow \mathbb{R}^n$ preserves the length cross ratios if there exists $u : V \rightarrow \mathbb{R}$ satisfying

$$\langle \dot{f}_j - \dot{f}_i, f_j - f_i \rangle = \frac{u_j + u_i}{2} |f_j - f_i|^2.$$

Such an infinitesimal deformation is called conformal. It is isometric if $u \equiv 0$.

As a remark, the vertex scaling are good *intrinsic* parameters to describe conformal deformations for triangulated spheres in \mathbb{R}^3 .

Proposition 3.6. Given an infinitesimally rigid triangulated sphere in \mathbb{R}^3 and a function $u : V \rightarrow \mathbb{R}$, there exists an infinitesimal conformal deformation unique up to Euclidean motions with vertex scaling u .

Proof. For every triangulated sphere in \mathbb{R}^3 , the space of infinitesimal conformal deformation C , including Euclidean motions, is of dimension at least $|V| + 6$. The map from the space of infinitesimal conformal deformations to the space of infinitesimal scale factors is a linear map $T : C \rightarrow \mathbb{R}^{|V|}$.

If the surface is infinitesimally rigid, the map T must be surjective and the space C is of dimension exactly $|V| + 6$. Otherwise, $\text{Ker}(T) > 6$ and there exists a non-trivial infinitesimal isometric deformation which leads to a contradiction. \square

Proposition 3.7 (Gluck [7]). Almost all simply connected closed triangulated surfaces in \mathbb{R}^3 are infinitesimally rigid.

Corollary 3.8. For almost all simply connected closed surfaces in \mathbb{R}^3 , there exists an infinitesimal conformal deformation unique up to Euclidean motions given any function $u : V \rightarrow \mathbb{R}$ as scale factors.

4. INFINITESIMAL ISOMETRIC DEFORMATIONS OF S^n IN \mathbb{R}^{n+1}

We show that every infinitesimal conformal deformation corresponds to an infinitesimal isometric deformation via stereographic projection. Therefore one can apply techniques from the theory of infinitesimal rigidity to that of infinitesimal conformal deformations, such as rigidity matrices.

For an inscribed triangulated sphere, we first establish a correspondence between its infinitesimal *isometric* deformations in \mathbb{R}^{n+1} and its infinitesimal *conformal* deformations tangent to the round sphere S^n .

Proposition 4.1. *Given an inscribed triangulated surface $f : V \rightarrow S^n \subset \mathbb{R}^{n+1}$, its space of infinitesimal isometric deformations in \mathbb{R}^{n+1} is isomorphic to the space of infinitesimal conformal deformations tangent to the sphere S^n .*

Proof. Suppose $v : V \rightarrow \mathbb{R}^{n+1}$ is an infinitesimal isometric deformation of f . Then its projection v^T to the tangent space of the sphere is

$$v^T := v - \langle v, f \rangle f.$$

Since $\langle v_j - v_i, f_j - f_i \rangle = 0$ we have

$$\begin{aligned} & \langle v_j^T - v_i^T, f_j - f_i \rangle \\ &= -\langle \langle v_j, f_j \rangle f_j - \langle v_i, f_i \rangle f_i, f_j - f_i \rangle \\ &= -\frac{1}{2} \langle (\langle v_j, f_j \rangle + \langle v_i, f_i \rangle)(f_j - f_i) + (\langle v_j, f_j \rangle - \langle v_i, f_i \rangle)(f_j + f_i), f_j - f_i \rangle \\ &= -\frac{1}{2} \langle (\langle v_j, f_j \rangle + \langle v_i, f_i \rangle) |f_j - f_i|^2 \rangle. \end{aligned}$$

Hence v^T is an infinitesimal conformal deformation with scaling factor $-\langle v, f \rangle$.

We are going to show that such a projection from infinitesimal isometric deformations to infinitesimal conformal deformations is bijective. Assume $v^T \equiv 0$. Then, we have

$$v_i = a_i f_i$$

for some $a : V \rightarrow \mathbb{R}$. Because v is an infinitesimal isometric deformation, we have

$$a_i = \langle v_i, f_i \rangle = -\langle v_j, f_j \rangle = -a_j \quad \forall e_{ij} \in E.$$

Consider the three vertices of any triangle, such condition is satisfied if and only if $a \equiv 0$. Hence, $v \equiv 0$ and the projection is injective.

On the other hand, suppose w is an infinitesimal conformal deformation tangent to S^n with conformal scaling u . We define an infinitesimal deformation by

$$v := w - uf.$$

Then,

$$\begin{aligned} & \langle v_j - v_i, f_j - f_i \rangle \\ &= \langle w_j - w_i, f_j - f_i \rangle - \langle u_j f_j - u_i f_i, f_j - f_i \rangle \\ &= \frac{u_i + u_j}{2} |f_j - f_i|^2 - \frac{1}{2} \langle (u_j - u_i)(f_j + f_i) + (u_j + u_i)(f_j - f_i), f_j - f_i \rangle \\ &= 0 \end{aligned}$$

which implies v is an infinitesimal isometric deformation and $v^T = w$. Hence the projection is bijective. \square

The following is the infinitesimal version of the Möbius invariance of conformal equivalence.

Lemma 4.2. *Let \dot{f} be an infinitesimal conformal deformation of $f : V \rightarrow \mathbb{R}^n$. Then for every Möbius transformations $\sigma : \mathbb{R}^n \cup \{\infty\} \rightarrow \mathbb{R}^n \cup \{\infty\}$, the infinitesimal deformation $d\sigma(\dot{f})$ of $\sigma \circ f$ is conformal.*

Proof. Since Möbius transformations are generated by Euclidean transformations and inversions, it suffices to consider the inversion under the unit sphere, $\sigma(f) = -f/|f|^2$. Suppose \dot{f} is an infinitesimal conformal deformation of f with infinitesimal scaling $u : V \rightarrow \mathbb{R}$. We have

$$d\sigma(\dot{f}) = -\frac{\dot{f}}{|f|^2} + \frac{2\langle \dot{f}, f \rangle}{|f|^4}.$$

By direct computation, we get

$$\begin{aligned} & \langle d\sigma(\dot{f}_j) - d\sigma(\dot{f}_i), \sigma(f_j) - \sigma(f_i) \rangle \\ &= (u_i - \frac{2\langle \dot{f}_i, f_i \rangle}{|f_i|^2} + u_j - \frac{2\langle \dot{f}_j, f_j \rangle}{|f_j|^2}) |\sigma(f_j) - \sigma(f_i)|^2 \end{aligned}$$

which implies $d\sigma(\dot{f})$ is an infinitesimal conformal deformation of $\sigma(f)$. \square

Proof of Theorem 1.2. With the fact that the stereographic projection is a Möbius transformation, Theorem 1.2 follows from Lemma 4.1 and Lemma 4.2. \square

5. INFINITESIMAL CONFORMAL DEFORMATIONS IN \mathbb{R}^3 (PROOF OF THEOREM 1.3)

In the following sections, we focus on infinitesimal conformal deformations of triangulated surfaces in Euclidean space. We establish an equation relating the infinitesimal vertex scaling and the change in dihedral angles.

We are going to prove Theorem 1.3 by considering moving frames on triangulated surfaces using quaternions. Every non-degenerate realization $f : V \rightarrow \mathbb{R}^3$ of a triangulated surface induces a frame $(T, N, T \times N)$ on every oriented edge e_{ij} , where $T(e_{ij}) = (f_j - f_i)/|f_j - f_i|$ and $N(e_{ij})$ is the normal of the left face of e_{ij} . Given a quaternionic function λ defined on oriented edges, a new frame $(\tilde{T}, \tilde{N}, \tilde{T} \times \tilde{N})$ is obtained by multiplication with quaternions:

$$(\tilde{T}(e), \tilde{N}(e), \tilde{T}(e) \times \tilde{N}(e)) = \lambda(e)^{-1}(T(e), N(e), T(e) \times N(e))\lambda(e).$$

Here we regard \mathbb{R}^3 as the purely imaginary part of quaternions $\text{Im } \mathbb{H}$. Assuming the triangulated surface is simply connected, the frames obtained are the moving frames of a realization \tilde{f} satisfying

$$(3) \quad d\tilde{f} = \bar{\lambda}df\lambda$$

if and only if the quaternionic function λ fulfills the following three conditions:

- (a) For all oriented edges in the same face $\{ijk\}$, the rotation $y \mapsto \lambda(e)^{-1}y\lambda(e)$ takes the face normal $N(e_{ij}) = N(e_{jk}) = N(e_{ki}) = N_{ijk}$ to the same unit vector \tilde{N}_{ijk} .
- (b) For any face $\{ijk\}$, we have the closedness condition

$$0 = \bar{\lambda}(e_{ij})df(e_{ij})\lambda(e_{ij}) + \bar{\lambda}(e_{jk})df(e_{jk})\lambda(e_{jk}) + \bar{\lambda}(e_{ki})df(e_{ki})\lambda(e_{ki}).$$

- (c) For any pair of oriented edges $(e, -e)$, the stretch-rotations $y \mapsto \bar{\lambda}(\pm e)y\lambda(\pm e)$ take $df(e)$ to the same vector $d\tilde{f}(e)$.

Suppose a 1-parameter family of deformations f_t is given by $\lambda(t)$ with $\lambda(0) \equiv 1$. Differentiating (3) with respect to t yields

$$(4) \quad d\dot{f} = 2\text{Im}(\dot{\lambda})df$$

In terms of the cross product in \mathbb{R}^3 , we can write

$$d\dot{f} = 2\text{Re}(\dot{\lambda})df + 2df \times \text{Im } \dot{\lambda}.$$

We now derive the corresponding conditions on $\dot{\lambda}$ for the existence of infinitesimal deformation \dot{f} . We first use an Ansatz to split $\dot{\lambda}(e)$ uniquely into scalar, normal and tangential components

$$(5) \quad \dot{\lambda}(e) := \frac{\sigma(e)}{2} - \frac{\omega(e)}{2}N(e) - \frac{Y(e)}{2}.$$

For an infinitesimal conformal deformation with vertex scaling u , we have

$$\sigma(e_{ij}) = \sigma(e_{ji}) = \frac{u_i + u_j}{2}.$$

Consider the change in the normal vector $N(e)$ at $t = 0$

$$\begin{aligned} (\lambda(e)^{-1}N(e)\lambda(e))^\cdot &= \left(\frac{\overline{\lambda(e)}N(e)\lambda(e)}{|\lambda(e)|^2} \right)^\cdot \\ &= \overline{\dot{\lambda}(e)}N(e) + N(e)\dot{\lambda}(e) - (\dot{\lambda}(e) + \overline{\dot{\lambda}(e)})N(e) \\ &= Y(e) \times N(e). \end{aligned}$$

Condition (a) implies for any face $\{ijk\}$, there exists Y_{ijk} such that

$$Y_{ijk} := Y(e_{ij}) = Y(e_{jk}) = Y(e_{ki}).$$

Furthermore, on a triangular face $\{ijk\}$ consisting of oriented edges e_{ij}, e_{jk}, e_{ki} , differentiating the equation in (b) yields

$$0 = \text{Im} (df(e_{ij})\dot{\lambda}(e_{ij}) + df(e_{jk})\dot{\lambda}(e_{jk}) + df(e_{ki})\dot{\lambda}(e_{ki}))$$

which is equivalent to

$$(6) \quad \begin{aligned} 0 = & \sigma(e_{ij})df(e_{ij}) + \sigma(e_{jk})df(e_{jk}) + \sigma(e_{ki})df(e_{ki}) \\ & + \omega(e_{ij})N_{ijk} \times df(e_{ij}) + \omega(e_{jk})N_{ijk} \times df(e_{jk}) + \omega(e_{ki})N_{ijk} \times df(e_{ki}). \end{aligned}$$

Notice that $df(e_{ij}) \in \text{span}\{N_{ijk} \times df(e_{jk}), N_{ijk} \times df(e_{ki})\}$. In particular

$$df(e_{ij}) = \cot(\angle jk)N_{ijk} \times df(e_{ki}) - \cot(\angle ki)N_{ijk} \times df(e_{jk}).$$

Substituting them into (6) implies

$$0 = \sum (\omega(e_{ij}) + (\sigma(e_{jk}) - \sigma(e_{ki})) \cot \angle jki) N_{ijk} \times df(e_{ij}).$$

Since $N_{ijk} \times df(e_{ij}), N_{ijk} \times df(e_{jk})$ and $N_{ijk} \times df(e_{ki})$ span an affine plane and

$$N_{ijk} \times df(e_{ij}) + N_{ijk} \times df(e_{jk}) + N_{ijk} \times df(e_{ki}) = 0,$$

there exists a unique number ω_{ijk} such that

$$\begin{aligned} \omega_{ijk} &= \omega(e_{ij}) + (\sigma(e_{jk}) - \sigma(e_{ki})) \cot \angle jki \\ &= \omega(e_{jk}) + (\sigma(e_{ki}) - \sigma(e_{ij})) \cot \angle kji \\ &= \omega(e_{ki}) + (\sigma(e_{ij}) - \sigma(e_{jk})) \cot \angle iki. \end{aligned}$$

As $\sigma(e_{ij}) = \frac{u_i + u_j}{2}$ we have

$$\omega(e_{ij}) = \omega_{ijk} - \frac{\cot \angle jki}{2}(u_j - u_i).$$

Therefore, in order to fulfill condition (a) and (b), $\dot{\lambda}$ has to be in the form

$$\dot{\lambda}(e_{ij}) = \frac{u_i + u_j}{4} - \frac{1}{2}(\omega_{ijk} - \frac{\cot \angle jki}{2}(u_j - u_i))N_{ijk} - \frac{Y_{ijk}}{2}.$$

where $u : V \rightarrow \mathbb{R}$, $\omega : F \rightarrow \mathbb{R}$ and $Y : F \rightarrow \mathbb{R}^3$ with $Y \perp N$. We define on each face

$$Z_{ijk} = -(\omega_{ijk}N_{ijk} + Y_{ijk})$$

and $\dot{\lambda}(e_{ij})$ becomes

$$(7) \quad \dot{\lambda}(e_{ij}) = \frac{u_i + u_j}{4} + \frac{Z_{ijk}}{2} + \frac{\cot \angle jki}{4}(u_j - u_i)N_{ijk}.$$

On the other hand condition (c) demands that

$$\begin{aligned}\bar{\lambda}(e)df(e) + df(e)\dot{\lambda}(e) &= \bar{\lambda}(-e)df(e) + df(e)\dot{\lambda}(-e) \\ \sigma(e)df(e) + df(e) \times \text{Im}(\dot{\lambda}(e)) &= \sigma(-e)df(e) + df(e) \times \text{Im}(\dot{\lambda}(-e))\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sigma(e) &= \sigma(-e) \\ df(e_{ij}) // \text{Im}(\dot{\lambda}(e)) - \text{Im}(\dot{\lambda}(-e)).\end{aligned}$$

Hence, on every oriented edge e , there exists a unique real number $\delta_e = \delta_{-e}$ such that

$$\dot{\lambda}(e) - \dot{\lambda}(-e) = \text{Im}(\dot{\lambda}(e)) - \text{Im}(\dot{\lambda}(-e)) = \delta_e T(e).$$

Lemma 5.1. *If an infinitesimal deformation is induced by $\dot{\lambda}$ in (7), then the change in the dihedral angle $\dot{\alpha}$ at edge $\{ij\}$ is*

$$\dot{\alpha}_{ij} = \langle Z_{ijk} - Z_{jil}, T(e_{ij}) \rangle$$

where $T(e_{ij}) = (f_j - f_i)/|f_j - f_i|$ and $\{ijk\}, \{jil\}$ are the left and the right face of e_{ij} .

Proof. Notice that

$$\sin \alpha_{ij} = \langle N_{ijk} \times N_{jil}, T(e_{ij}) \rangle.$$

Differentiating both sides yields

$$\begin{aligned}\dot{\alpha}_{ij} \cos \alpha_{ij} &= \langle \dot{N}_{ijk} \times N_{jil} + N_{ijk} \times \dot{N}_{jil}, T(e_{ij}) \rangle \\ &= \langle (N_{ijk} \times Z_{ijk}) \times N_{ilj} + N_{ijk} \times (N_{ilj} \times Z_{jil}), T(e_{ij}) \rangle \\ &= \cos \alpha \langle Z_{ijk} - Z_{jil}, T(e_{ij}) \rangle.\end{aligned}$$

Hence

$$\dot{\alpha}_{ij} = \langle Z_{ijk} - Z_{jil}, T(e_{ij}) \rangle.$$

□

Therefore condition (c) becomes

$$(8) \quad Z_{ijk} - Z_{ilj} = -(u_j - u_i) \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle ilj}{2} N_{ilj} \right) + \dot{\alpha}_{ij} T(e_{ij}).$$

which is equivalent to

$$\begin{cases} \langle df(e_{ij}), dZ(e_{ij}^*) \rangle = \dot{\alpha}_{ij} |df(e_{ij})| \\ -df(e_{ij}) \times dZ(e_{ij}^*) + \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle ilj}{2} N_{ilj} \right) \times df(e_{ij}) du(e_{ij}) = 0 \end{cases}$$

as stated in Theorem 1.3.

Conversely, given $Z : F \rightarrow \mathbb{R}^3$ satisfying Equation (8) for some $u : V \rightarrow \mathbb{R}$, the 1-form

$$\begin{aligned}\eta(e_{ij}) &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times \left(Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk} \right) \\ &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times \left(Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil} \right)\end{aligned}$$

is well defined and is closed. If the triangulated surface is simply connected, then there exists an infinitesimal conformal deformation $\dot{f} : V \rightarrow \mathbb{R}^3$ unique up to translations such that

$$d\dot{f} = \eta.$$

6. ANGULAR VELOCITY EQUATION

The derivation in the previous section yields an interesting equation.

Theorem 6.1 (Angular velocity equation). *Under an infinitesimal conformal deformation with scaling factor $u : V \rightarrow \mathbb{R}$, the change in dihedral angles $\dot{\alpha}$ satisfy for every interior vertex i*

$$\sum_j \dot{\alpha}_{ij} T_{e_{ij}} = \sum_j (u_j - u_i) \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle ilj}{2} N_{ilj} \right) = \sum \dot{\beta}_{ijk} N_{ijk}.$$

where $\dot{\beta}_{ijk}$ denotes the change in $\angle ijk$.

Proof. It follows from Equation (8) that for every interior vertex i

$$\sum_j (\dot{\alpha}_{ij} T_{e_{ij}} - (u_j - u_i) \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle ilj}{2} N_{ilj} \right)) = \sum_j (Z_{ijk} - Z_{ilj}) = 0$$

Furthermore

$$\cos \beta_{kij} = \frac{\langle df(e_{ij}), df(e_{ik}) \rangle}{|df(e_{ij})| |df(e_{ik})|}.$$

Differentiating both sides yields

$$-\dot{\beta}_{kij} \sin \beta_{kij} = -\sin \beta_{kij} \left((u_j - u_i) \frac{\cot \angle jki}{2} + (u_k - u_i) \frac{\cot \angle ilj}{2} \right)$$

and thus

$$\dot{\beta}_{kij} = (u_j - u_i) \frac{\cot \angle jki}{2} + (u_k - u_i) \frac{\cot \angle ilj}{2}.$$

□

For every infinitesimal isometric deformation, we have $u \equiv 0$ and hence obtain the standard identity [17, Lemma 28.2]

$$\sum_j \dot{\alpha}_{ij} T_{e_{ij}} = 0 \quad \forall i \in V_{int}$$

which is related to an infinitesimal deformation of spherical polygon with fixed edge lengths [22].

Corollary 6.2 ([24, 14]). *Suppose an infinitesimal conformal deformation of an immersed triangulated surface in the plane has scaling factor $u : V \rightarrow \mathbb{R}$. Then the function u is a discrete harmonic function with respect to the cotangent Laplacian, i.e. for every interior vertex i*

$$\sum_j (\cot \angle jki + \cot \angle ilj) (u_j - u_i) = 0$$

and the change in dihedral angle $\dot{\alpha}$ satisfies

$$\sum_j \dot{\alpha}_{ij} T_{e_{ij}} = 0.$$

7. DISCRETE DIRAC OPERATOR

We are interested in parameterizing the space of infinitesimal conformal deformations in terms of extrinsic geometry. Under infinitesimal conformal deformations, we consider the *change in mean curvature half-density* $\rho : V_{int} \rightarrow \mathbb{R}$

$$\rho_i := \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|.$$

We first demonstrate its relation to Schläfli formula.

Proposition 7.1. (*Scalar Schläfli Formula*) *For every infinitesimal conformal deformation of a closed triangulated surface $f : V \rightarrow \mathbb{R}^3$, we have*

$$\sum_i \rho_i = \sum_{ij} \dot{\alpha}_{ij} |df(e_{ij})| = 0.$$

where $\dot{\alpha}$ is the change in the dihedral angles.

Proof. Given an infinitesimal conformal deformation, Theorem 1.3 shows that there exists $(u, Z) \in \mathbb{R}^V \times \mathbb{R}^{3F}$ such that

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})| = \frac{1}{2} \sum_j \langle df(e_{ij}), Z_{ijk} - Z_{jki} \rangle.$$

Hence we have

$$\sum_i \rho_i = \sum_{ij \in E} \dot{\alpha}_{ij} |df(e_{ij})| = \sum_{ijk} \langle df(e_{ij}) + df(e_{ij}) + df(e_{ij}), Z_{ijk} \rangle = 0.$$

□

Remark 7.2. *Scalar Schläfli Formula holds for general infinitesimal deformations of closed triangulated surfaces [17].*

A heuristic argument of the degrees of freedom indicate that the change in mean curvature half-density is a good candidate to parametrize infinitesimal conformal deformations extrinsically. Notice that the space of infinitesimal conformal deformations of a triangulated sphere is at least $3V - (E - V) = V + 6$. Euclidean motions and similarity are trivial conformal deformations that preserve dihedral angles. Hence it is reasonable to ask if, given a function $\rho : V \rightarrow \mathbb{R}$ with $\sum_i \rho_i = 0$, does there exist an infinitesimal conformal deformation unique up to Euclidean motions and similarity with ρ as the change in mean curvature half-density? The answer is positive and is provided in Theorem 1.4. We follow the proof in the smooth theory [19] by developing the discrete Dirac operator D for closed triangulated surfaces.

We rewrite Theorem 1.3 in terms of discrete differential forms. We denote

$$\begin{aligned} dZ(e_{ij}^*) &:= Z_{ijk} - Z_{jil} \\ df(e_{ij}^*) &:= \left(\frac{\cot \angle jki}{2} N_{ijk} + \frac{\cot \angle jil}{2} N_{jil} \right) \times df(e_{ij}) \end{aligned}$$

where $\{ijk\}, \{jil\}$ are the left face and the right face of e_{ij} . Furthermore $df(e_{ij}^*)$ is the vector from the circumcenter of the right face $\{jil\}$ to that of the left face $\{ijk\}$. Suppose an infinitesimal conformal deformation is given by $(u, Z) \in \mathbb{R}^{V+3F}$ as in Theorem 1.3, then we have

$$\begin{aligned} \langle df(e_{ij}), dZ(e_{ij}^*) \rangle &= \dot{\alpha}_{ij} |df(e_{ij})|, \\ -df(e_{ij}) \times dZ(e_{ij}^*) + df(e_{ij}^*) du(e_{ij}) &= 0 \end{aligned}$$

where $\dot{\alpha}$ is the change in the dihedral angles.

Definition 7.3. *We denote $\mathbb{R}^{V_{int}+2E_{int}}$ as the space of pairs (α, W) , where α is a real valued function defined on interior vertices and W is \mathbb{R}^3 -valued function on interior edges such that W_{ij} is perpendicular to $df(e_{ij})$ for every edge $\{ij\}$.*

Definition 7.4. *Given a realization of a triangulated surface $f : V \rightarrow \mathbb{R}^3$, we define the discrete Dirac operator*

$$\begin{aligned} D : \mathbb{R}^V \times \mathbb{R}^{3F} &\rightarrow \mathbb{R}^{V_{int}} \times \mathbb{R}^{2E_{int}} \\ (u, Z) &\mapsto (\rho, U) \end{aligned}$$

where

$$\begin{aligned}\rho_i &= \frac{1}{2} \sum_j \langle df(e_{ij}), dZ(e_{ij}^*) \rangle, \\ U_{ij} &= -df(e_{ij}) \times dZ(e_{ij}^*) + df(e_{ij}^*) du(e_{ij}).\end{aligned}$$

In particular, $U_{ij} \perp df(e_{ij})$.

We are going to derive the adjoint of the discrete Dirac operator on closed triangulated surfaces.

Definition 7.5. We define inner products on \mathbb{R}^{V+2E} and \mathbb{R}^{V+3F} . For $(\alpha, W), (\tilde{\alpha}, \tilde{W}) \in \mathbb{R}^{V+2E}$

$$((\alpha, W), (\tilde{\alpha}, \tilde{W})) := \sum_i \alpha_i \tilde{\alpha}_i + \sum_{ij} \langle W_{ij}, \tilde{W}_{ij} \rangle.$$

For $(\beta, Y), (\tilde{\beta}, \tilde{Y}) \in \mathbb{R}^{V+3F}$

$$((\beta, Y), (\tilde{\beta}, \tilde{Y})) := \sum_i \beta_i \tilde{\beta}_i + \sum_{ijk} \langle Y_{ijk}, \tilde{Y}_{ijk} \rangle.$$

With the inner products, we know that the adjoint D^* of the discrete Dirac operator is a map $D^* : \mathbb{R}^{V+2E} \rightarrow \mathbb{R}^{V+3F}$ satisfying

$$((u, Z), D^*(\alpha, W)) = (D(u, Z), (\alpha, W))$$

for $(u, Z) \in \mathbb{R}^{V+3F}$ and $(\alpha, W) \in \mathbb{R}^{V+2E}$.

Proposition 7.6. Given a realization $f : V \rightarrow \mathbb{R}^3$ of a closed triangulated surface, the adjoint of the discrete Dirac operator is

$$\begin{aligned}D^* : \mathbb{R}^V \times \mathbb{R}^{2E} &\rightarrow \mathbb{R}^V \times \mathbb{H}^{3F} \\ (\alpha, W) &\mapsto (\tilde{\rho}, Y)\end{aligned}$$

where

$$\begin{aligned}\tilde{\rho}_i &= - \sum_{ij \in E} \langle df(e_{ij}^*), W_{ij} \rangle \\ Y_{ijk} &= df(e_{ij}) \times W_{ij} + df(e_{jk}) \times W_{jk} + df(e_{ki}) \times W_{ki} \\ &\quad + \frac{\alpha_i + \alpha_j}{2} df(e_{ij}) + \frac{\alpha_j + \alpha_k}{2} df(e_{jk}) + \frac{\alpha_k + \alpha_i}{2} df(e_{ki})\end{aligned}$$

and $df(e_{ij}^*)$ is the vector from the circumcenter of the right face of e_{ij} to that of the left face.

Proof. Direct computations yield

$$\begin{aligned}((u, Z), D^*(\alpha, 0)) &= \sum_{i \in V} \frac{\alpha_i}{2} \sum_j \langle df(e_{ij}), dZ(e_{ij}^*) \rangle \\ &= \sum_{ijk} \langle Z_{ijk}, \frac{\alpha_i + \alpha_j}{2} df(e_{ij}) + \frac{\alpha_j + \alpha_k}{2} df(e_{jk}) + \frac{\alpha_k + \alpha_i}{2} df(e_{ki}) \rangle, \\ ((u, 0), D^*(0, W)) &= \sum_{ij \in E} \langle df(e_{ij}^*) du(e_{ij}), W_{ij} \rangle \\ &= - \sum_{i \in V} u_i \sum_j \langle df(e_{ij}^*), W_{ij} \rangle, \\ ((0, Z), D^*(0, W)) &= \sum_{ij \in E} \langle -df(e_{ij}) \times dZ(e_{ij}^*), W_{ij} \rangle \\ &= \sum_{ijk} -\langle Z_{ijk}, W_{ij} \times df(e_{ij}) + W_{jk} \times df(e_{jk}) + W_{ki} \times df(e_{ki}) \rangle\end{aligned}$$

By linearity of the discrete Dirac operator D^* , we obtain the formula as stated. \square

With elementary linear algebra, we have the following.

Lemma 7.7. *On a closed triangulated surfaces, we have $3|E| = 2|F|$ and*

$$\begin{aligned} \dim(\text{Im } D) &= \dim(\text{Im } D^*), \\ \dim(\text{Ker } D) &= \dim(\text{Ker } D^*), \\ (\text{Ker } D^*)^\perp &= \text{Im } D. \end{aligned}$$

In the following, given any vector U_{ij} on edge $\{ij\}$, we write U_{ij}^\dagger as the component orthogonal to $df(e_{ij})$.

Lemma 7.8.

$$\{(\alpha, W^\dagger) \in \mathbb{R}^V \times \mathbb{R}^{2E} \mid \alpha \in \mathbb{R}, W \in \mathbb{R}^3\} \subset \text{Ker } D^*$$

and hence $\dim(\text{Ker } D^*) \geq 4$.

Proof. Let α be a constant real valued function on vertices, W be a constant \mathbb{R}^3 -vector on edges. We write $(\tilde{\rho}, Y) := D^*(\alpha, W^\dagger)$ and have

$$\begin{aligned} \tilde{\rho}_i &= - \sum_j \langle df(e_{ij}^*), W \rangle = 0, \\ Y_{ijk} &= df(e_{ij}) \times W + df(e_{jk}) \times W + df(e_{ki}) \times W \\ &\quad + \frac{\alpha_i + \alpha_j}{2} df(e_{ij}) + \frac{\alpha_j + \alpha_k}{2} df(e_{jk}) + \frac{\alpha_k + \alpha_i}{2} df(e_{ki}) \\ &= 0. \end{aligned}$$

If $W^\dagger \equiv 0$ for a constant vector W , it implies W is parallel to all edges of f . Since f is non-degenerate, we have $W \equiv 0$. Hence, the nullity of D^* is at least 4. \square

Lemma 7.9. *If $\dim(\text{Ker } D) = 4$, then for all $(\rho, U) \in \mathbb{R}^V \times \mathbb{R}^{2E}$,*

$$\sum_{i \in V} \rho_i = 0 \text{ and } \sum_{ij \in E} U_{ij} = 0 \Leftrightarrow (\rho, U) \in \text{Im } D.$$

Proof. Suppose there exists $(u, Z) \in \mathbb{R}^{V+3F}$ such that $D(u, Z) = (\rho, U)$. Then Proposition 7.1 implies $\sum_{i \in V} \rho_i = 0$ while $\sum_{ij \in E} U_{ij} = 0$ follows from direct computation.

Conversely, the assumption $\dim(\text{Ker } D^*) = 4$ which means $\text{Ker } D^*$ contains constant functions only. Therefore,

$$\sum_{i \in V} \rho = 0 \text{ and } \sum_{ij \in E} U_{ij} = 0$$

implies $(\rho, U) \in (\text{Ker } D^*)^\perp = \text{Im } D$. \square

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Notice that infinitesimal translations correspond to $u \equiv 0, Z \equiv 0$ in Theorem 1.3. Furthermore infinitesimal rotations are given by $u \equiv 0, Z \equiv \text{const.}$ while uniform scaling by $u \equiv \text{const.}, Z \equiv 0$.

Given a function $\rho : V \rightarrow \mathbb{R}^3$ with $\sum_i \rho_i = 0$, then $(\rho, 0) \in \text{Im } D$ and hence there exists $(u, Z) \in \mathbb{R}^{V+3F}$ such that $D(u, Z) = (\rho, 0)$. Since $\dim \ker D = 4$, the functions u, Z are unique up to constants. Thus with Theorem 1.3, we conclude that there exists an infinitesimal conformal deformation unique up to similarities satisfying

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|.$$

\square

The corresponding Dirac operator [11] on closed surfaces in the smooth theory is self adjoint. Hence it is not surprising that there is an analogue of Theorem 1.3 in terms of D^* .

Proposition 7.10. *Let $\dot{f} : V \rightarrow \mathbb{R}^3$ be an infinitesimal conformal deformation of a non-degenerate realization $f : V \rightarrow \mathbb{R}$ with infinitesimal scaling $u : V \rightarrow \mathbb{R}$. Then there exists unique $W : E \rightarrow \mathbb{R}^3$ such that*

$$(9) \quad \begin{aligned} d\dot{f}(e_{ij}) &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times W_{ij}, \\ 0 &= \langle W_{ij}, df(e_{ij}) \rangle. \end{aligned}$$

Furthermore, we have $D^*(u, W) = (\rho, 0)$ where

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|$$

and $\dot{\alpha}$ is the change in dihedral angle.

Conversely, if the triangulated surface is simply connected and functions $u : V \rightarrow \mathbb{R}$, $W : E \rightarrow \mathbb{R}^3$ satisfy $D^*(u, W) = (\rho, 0)$ for some ρ , then there exists an infinitesimal conformal deformation \dot{f} unique up to translation given via (9).

Proof. Given an infinitesimal conformal deformations, the existence and uniqueness of W is immediate. For every face $\{ijk\}$, we have

$$\begin{aligned} 0 = d\dot{f}(e_{ij}) + d\dot{f}(e_{jk}) + d\dot{f}(e_{ki}) &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times W_{ij} \\ &\quad + \frac{u_j + u_k}{2} df(e_{jk}) + df(e_{jk}) \times W_{jk} \\ &\quad + \frac{u_k + u_i}{2} df(e_{ki}) + df(e_{ki}) \times W_{ki}. \end{aligned}$$

Hence $D^*(u, W) = (\rho, 0)$ for some $\rho : V_{int} \rightarrow \mathbb{R}$. On the other hand, by Theorem 1.3, there exists an $Z \rightarrow \mathbb{R}^3$ such that for every face $\{ijk\}$

$$\begin{aligned} d\dot{f}(e_{ij}) &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times (Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk}) \\ &= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times (Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil}) \end{aligned}$$

and

$$(Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk}) - (Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil}) = \dot{\alpha}_{ij} T(e_{ij}).$$

Thus we have

$$W_{ij} = (Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk})^\dagger = (Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil})^\dagger = W_{ji}$$

where \dagger denotes the component orthogonal to $df(e_{ij})$. We denote the circumcenter of $\{ijk\}$ as f_{ijk} . We have

$$\begin{aligned} \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})| &= \sum_j \langle \dot{\alpha}_{ij} T_{e_{ij}}, f_{ijk} - f_i \rangle \\ &= \sum_j \left(\langle Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk}, f_{ijk} - f_i \rangle \right. \\ &\quad \left. - \langle Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil}, f_{ijk} - f_i \rangle \right) \\ &= - \sum_j \langle Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil}, (f_{ijk} - f_i) - (f_{jil} - f_i) \rangle \\ &= - \sum_j \langle (Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil})^\dagger, df(e_{ij}^*) \rangle \end{aligned}$$

$$=\rho_i$$

□

The following explains the name of the discrete Dirac operator, which is the “square root” of the cotangent Laplacian.

Proposition 7.11. *For any real-valued function defined on vertices $\alpha : V \rightarrow \mathbb{R}$, we have $D \frac{1}{A} D^*(\alpha, 0) = (\rho, U)$ where*

$$\rho_i = - \sum_j (\cot \angle jki + \cot \angle jil)(\alpha_j - \alpha_i),$$

$$U_{ij} = (\alpha_j - \alpha_i)(N_{ijk} - N_{jil}).$$

Here $A : F \rightarrow \mathbb{R}$ is the signed area of the corresponding triangle under the realization and $\{ijk\}, \{jil\} \in F$ are the left and the right faces of e_{ij} .

Proof. Notice that

$$\begin{aligned} D^*(\alpha, 0)_{ijk} &= - \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{2} \\ (D \frac{1}{A} D^*(\alpha, 0))_i &= \sum_j \langle df(e_{ij}), \frac{1}{A_{ijk}} D^*(\alpha, 0)_{ijk} - \frac{1}{A_{jil}} D^*(\alpha, 0)_{jil} \rangle \\ &= \sum_j \langle df(e_{ij}), \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{2A_{ijk}} \\ &\quad - \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{2A_{jil}} \rangle \\ &= - \sum_j (\cot \angle jki + \cot \angle jil)(\alpha_j - \alpha_i) \end{aligned}$$

where A_{ijk} is the signed area of triangle $\{ijk\}$. On the other hand

$$\begin{aligned} U_{ij} &= df(e_{ij}) \times \left(- \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{2A_{ijk}} \right. \\ &\quad \left. + \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{2A_{jil}} \right) \\ &= (\alpha_j - \alpha_i)(N_{ijk} - N_{jil}). \end{aligned}$$

□

8. EXAMPLES

We restrict our attention here to triangulated surfaces with vertices on the unit sphere. We investigate if they possess infinitesimal conformal deformations where the change in mean curvature half-density vanishes.

Proposition 8.1 ([15]). *Every infinitesimal isometric deformation of a triangulated surface inscribed on a sphere satisfies for every interior vertex i*

$$\sum_j \dot{\alpha}_{ij} |df(e_{ij})| = 0.$$

Example 8.2. *Jessen’s orthogonal icosahedron (Figure 2) is obtained from a regular icosahedron by flipping 6 edges symmetrically [10, 8]. Its vertices are the same as the regular icosahedron and hence can be circumscribed in a sphere. It is known to be infinitesimally flexible. By Proposition 8.1 the change in mean curvature half-density vanishes. Hence $\dim(\text{Ker } D) > 4$.*

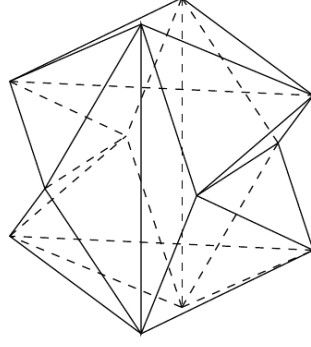


FIGURE 2. Jessen's orthogonal icosahedron

Example 8.3. Figure 3 shows one of Bricard's octahedra inscribed in a sphere. It is flexible and therefore Proposition 8.1 implies $\dim(\text{Ker } D) > 4$.

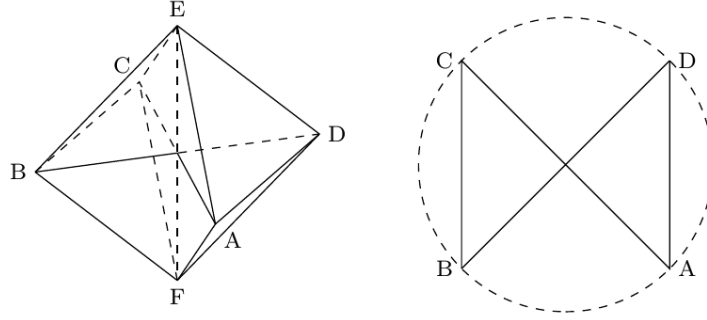


FIGURE 3. Bricard's octahedron and its top view

Dehn's theorem states that convex polyhedra are infinitesimally rigid. It implies there is no infinitesimal isometric deformation. However, one could still ask if there is any infinitesimal conformal deformation where the change in mean curvature half-density vanishes. For inscribed polyhedra, their infinitesimal conformal deformations are exactly given by normal deformations up to isometric deformations:

Proposition 8.4. Suppose $f : V \rightarrow S^2$ is an inscribed triangulated surface. For every $u : V \rightarrow \mathbb{R}$, the infinitesimal deformation $\dot{f} = uf$ is conformal with scaling u and the change in mean curvature half-density is

$$\sum_j \dot{\alpha}_{ij} |df(e_{ij})| = - \sum_j (u_j - u_i) \left(\frac{\cos d_{ijk} \cot \angle jki}{2} + \frac{\cos d_{jil} \cot \angle ilj}{2} \right) \quad \forall i \in V_{int}$$

where d_{ijk} is the distance from the origin to the face $\{ijk\}$.

Proof. Let $u : V \rightarrow \mathbb{R}$ and $\dot{f} = uf$. We have

$$\langle \dot{f}_j - \dot{f}_i, f_j - f_i \rangle = u_j + u_i - (u_j + u_i) \langle f_j, f_i \rangle = \frac{u_j + u_i}{2} |f_j - f_i|^2.$$

and hence \dot{f} is an infinitesimal conformal deformation with scaling u . In terms of Theorem 1.3, such a deformation is given by $Z : F \rightarrow \mathbb{R}^3$

$$Z_{ijk} = -\frac{\cos d_{ijk}}{4A_{ijk}} (u_i df(e_{jk}) + u_j df(e_{ki}) + u_k df(e_{ij}))$$

where A_{ijk} is the area of the triangle $\{ijk\}$. For every interior vertex i , the change in mean curvature half-density is

$$\begin{aligned} \sum_j \dot{\alpha}_{ij} |df(e_{ij})| &= \sum_j \langle df(e_{ij}), dZ(e_{ij}^*) \rangle \\ &= - \sum_j (u_j - u_i) \left(\frac{\cos d_{ijk} \cot \angle jki}{2} + \frac{\cos d_{jil} \cot \angle ilj}{2} \right) \end{aligned}$$

□

Example 8.5. *The regular octahedron has $\dim(\text{Ker } D) = 4$, that means it does not possess non-trivial infinitesimal conformal deformation with vanishing change in mean curvature half-density. On one hand, all its infinitesimal isometric deformations are trivial since it is convex. On the other hand, for each edge $\{ij\}$, the coefficient $(\cos d_{ijk} \cot \angle jki + \cos d_{jil} \cot \angle ilj)/2$ is strictly positive. If there is an infinitesimal conformal deformation such that $\sum_j \dot{\alpha} |df(e_{ij})| = 0$ for all vertices, then the conformal scaling u must be constant and the deformation is induced by a uniform scaling.*

Applying Gluck's argument [7, Theorem 6.1] to the discrete Dirac operator, the set of non-degenerate triangulated spheres in \mathbb{R}^3 with $\dim \text{Ker } D = 4$ is the complement of a real algebraic variety in $\mathbb{R}^{3|V|}$. Hence the set is open and dense as long as the algebraic variety is proper.

Corollary 8.6. *For almost all octahedra in \mathbb{R}^3 , given any $\rho : V \rightarrow \mathbb{R}$ with $\sum_i \rho_i = 0$, there exists an infinitesimal conformal deformation unique up to Euclidean motion and similarity such that*

$$\rho_i = \frac{1}{2} \sum_j \dot{\alpha}_{ij} |df(e_{ij})|$$

where $\dot{\alpha}$ is the change in dihedral angles.

9. TRIANGULATED SURFACES OF HIGH GENUS

Lemma 9.1. *For a closed triangulated surface of genus g , there exists $2g$ closed dual 1-forms $\omega_1, \omega_2, \dots, \omega_{2g} : \vec{E} \rightarrow \mathbb{R}$ such that a closed primal 1-form $\eta : \vec{E} \rightarrow \mathbb{R}$ is exact if and only if for $k = 1, 2, \dots, 2g$*

$$\sum_{ij \in E} \omega_k(e_{ij}^*) \eta(e_{ij}) = 0.$$

The dual 1-forms ω_i are called *harmonic forms* and the lemma follows from the Hodge decomposition of discrete differential forms [6].

Theorem 9.2. *Suppose $f : M \rightarrow \mathbb{R}^3$ is a realization of a closed triangulated surface of genus g with $\dim(\text{Ker } D) = 4$. Let $\omega_1, \dots, \omega_{2g}$ form a basis of harmonic 1-forms and $\mathbf{e}_1 := (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$ be the basis of \mathbb{R}^3 . Then for every $k = 1, 2, \dots, 2g$ and $l = 1, 2, 3$, there exists $(u_{kl}, Z_{kl}) \in \mathbb{R}^{V+3F}$ unique up to constants such that $D(u_{kl}, Z_{kl}) = (\alpha_{kl}, W_{kl}) \in \mathbb{R}^{V+2E} \subset \mathbb{R}^{V+3E}$ where*

$$\begin{aligned} \alpha_{kl,i} &= \sum_j \langle \mathbf{e}_l, \omega_k(e_{ij}^*) df(e_{ij}) \rangle, \\ W_{kl,ij} &= \mathbf{e}_l \times \omega_k(e_{ij}^*) df(e_{ij}). \end{aligned}$$

Furthermore, given $(u, Z) \in \mathbb{R}^{V+3F}$ with $D(u, Z) = (\rho, 0)$, the \mathbb{R}^3 -valued primal 1-form

$$\eta(e_{ij}) = \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times (Z_{ijk} + \frac{\cot \angle jki}{2} (u_j - u_i) N_{ijk})$$

$$= \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times (Z_{jil} + \frac{\cot \angle jil}{2} (u_i - u_j) N_{jil})$$

is exact if and only if for every $k = 1, 2, \dots, 2g$ and $l = 1, 2, 3$,

$$\sum_i \rho_i u_{kl,i} = 0.$$

Proof. Since ω_k is a closed dual 1-form, we know it satisfies for every vertex i

$$\sum_j \omega_k(e_{ij}^*) = 0$$

and hence

$$\sum_{i \in V} \sum_{ij \in E:i} \langle \mathbf{e}_l, \omega_k(e_{ij}^*) df(e_{ij}) \rangle = -2 \sum_{i \in V} (\langle \mathbf{e}_l, f_i \rangle \sum_{ij \in E:i} \omega_k(e_{ij}^*)) = 0.$$

On the other hand,

$$\sum_{ij \in E} \mathbf{e}_l \times \omega_k(e_{ij}^*) df(e_{ij}) = - \sum_{i \in V} (\mathbf{e}_l \times f_i \sum_{ij \in E:i} \omega_k(e_{ij}^*)) = 0.$$

By Lemma (7.9), the sums being zero imply the existence of (u_{kl}, Z_{kl}) as claimed.

Given $(u, Z) \in \mathbb{R}^{V+3F}$ with $D(u, Z) = (\rho, 0)$, the 1-form η is closed by Theorem 1.3. We write $\eta(e_{ij}) = \frac{u_i + u_j}{2} df(e_{ij}) + df(e_{ij}) \times W_{ij}$ where $W_{ij} \perp df(e_{ij})$. Lemma 9.1 implies that η is exact if and only if for every $k = 1, 2, \dots, 2g$ and $l = 1, 2, 3$,

$$\begin{aligned} 0 &= \sum_{ij \in E} \omega(e_{ij}^*) \langle \eta(e_{ij}), \mathbf{e}_l \rangle \\ &= \sum_{i \in V} \sum_{ij \in E:i} \langle \mathbf{e}_l, \omega_k(e_{ij}^*) df(e_{ij}) \rangle u_i + \sum_{ij \in E} \langle \omega_k(e_{ij}^*) \mathbf{e}_l \times df(e_{ij}), W_{ij} \rangle \\ &= (D(u_{kl}, Z_{kl}), (u, W)) \\ &= ((u_{kl}, Z_{kl}), D^*(u, W)) \\ &= ((u_{kl}, Z_{kl}), (\rho, 0)) \\ &= \sum_{i \in V} \rho_i u_{kl,i}. \end{aligned}$$

□

10. DISCUSSION

Proposition 3.6 and Theorem 1.4 provide two ways to parametrize the infinitesimal conformal deformations of a triangulated surface in \mathbb{R}^3 . For infinitesimally rigid triangulated spheres, one can use the vertex scalings as intrinsic parameters. For triangulated spheres with $\dim \ker D = 4$, one can use the change in mean curvature half-density as extrinsic parameters. One might be tempting to think that both variables together are sufficient to describe the infinitesimal conformal deformations of an arbitrary triangulated surface. However, this is not true.

Definition 10.1. [15] *A realization $f : V \rightarrow \mathbb{R}$ of a triangulated surface is called isothermic if there exists $k : E \rightarrow \mathbb{R}$ such that for each interior vertex i*

$$\begin{aligned} \sum_j k_{ij} |f_j - f_i|^2 &= 0, \\ \sum_j k_{ij} (f_j - f_i) &= 0. \end{aligned}$$

Proposition 10.2. [15] *A simply connected surface is isothermic if and only if it admits a non-trivial infinitesimal isometric deformation such that the change in mean curvature half-density vanishes.*

As a remark: isothermic surfaces are singularities of the space of conformal realizations. They are analogues of infinitesimally flexible surfaces in the case of isometric realizations.

Proposition 10.3. [15] *The space of infinitesimal conformal deformations of a closed triangulated surface of genus g in \mathbb{R}^3 has dimension greater or equal to $|V| + 6 - 6g$. The inequality is strict if and only if the surface is isothermic.*

Proposition 10.4. [15] *The class of isothermic surfaces is Möbius invariant.*

Jessen's orthogonal icosahedron in Example 8.2 and Bricard's octahedron in Example 8.3 are isothermic surfaces.

For a simply connected surface in the smooth theory, $\dim(\text{Ker } D)$ is Möbius invariant [19, Lemma 26]. We conjecture that the discrete analogue of this statement holds as well:

Conjecture 10.5. *For a simply connected triangulated surface in space, the nullity of the discrete Dirac operator $\dim(\text{Ker } D)$ is invariant under Möbius transformations.*

It is interesting to know if we can extend Corollary 8.6 to triangulated spheres of general combinatorics as like as Gluck's theorem [7]. There are two important ingredients to Gluck's theorem. One is Steinitz' theorem that every triangulated sphere admits a strictly convex realization into \mathbb{R}^3 . The other is Dehn's theorem that all convex realizations are infinitesimally rigid.

Conjecture 10.6. *Every abstract triangulated sphere admits a realization into \mathbb{R}^3 with $\dim \text{Ker } D = 4$, which means it does not process a non-trivial infinitesimal conformal deformation with vanishing change in mean curvature half-density.*

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